MMP Learning Seminar Week 68

Boundedness of Moduli.

What we've done so far :

<u>Theorem 1.2</u> (X, Δ) log pair s.t. coeffs of $\Delta \in [0, 1] \cap \mathbb{Q}$.

$\pi : X \to U$ proj morphism s.t. U smooth, (X, Δ) log smooth over U.

If $\exists\, 0 \in U$ s.t. (X_0, Δ_0) has a good minimal model,

$\begin{cases} (X, \Delta) \text{ has good minimal model over } U \\ \text{All fibres have good minimal models} \end{cases}$

<u>Cor 1.3</u> (X, Δ) log pair s.t. Δ $\mathbb{Q}$-divisor

$X \to U$ flat proj morphism s.t. U smooth & Supp Δ

contain neither a component of a fibre nor a codim 1

component of the singular locus of a fibre

Then $U_0 = \{ u \in U \mid (X_u, \Delta_u) \text{ dlt \& has a good minimal model} \}$

is constructible.

Lemma 2.2.2 $f: X \to Z$ bir'l morphism b/t lc pairs (X, Δ) & (Z, B)

Suppose $K_X + \Delta$ big & (X, Δ) has a lc model $g: X \dashrightarrow Y$.

If $f_* \Delta \leq B$ and $\mathrm{vol}(X, K_X + \Delta) = \mathrm{vol}(Z, K_Z + B)$,

then the induced bir'l map $Z \dashrightarrow Y$ is the lc model of (Z, B).

Lemma 2.8.1 (X, Δ) log smooth pair. Suppose $\lfloor \Delta \rfloor = 0$.

Then $\exists\, \pi: Y \to X$ sequence of smooth blow ups of the strata of (X, Δ) such that if we write

$K_Y + T = \pi^*(K_X + \Delta) + E$ where $T \geq 0$ and $E \geq 0$ have no common components, $\pi_* T = \Delta$ and $\pi_* E = 0$,

then no two components of T intersect.

Cor 4.3 $\pi: X \to U$ be a projective contraction morphism to smooth variety U.

(X, Δ) log smooth over U & coeffs of $\Delta \leq 1$.

Then $\mathrm{vol}(X_u, K_{X_u} + \Delta_u)$ is independent of $u \in U$.

(Abbreviations)

[AL] : ACC for LCTs (Hacon – McKernan – Xu)

[BA] : On Bir'l Automorphisms of varieties of general type

(")

Lemma 7.1 $w \in \mathbb{R}^+$, $I \subset [0,1]$ DCC set.

Fix a log smooth pair (Z, B) where Z proj variety.

Let $\mathcal{F} := \{ (X, \Delta)$ log smooth $\mid \mathrm{vol}(X, K_X + \Delta) = w$

coeff's of $\Delta \subseteq I$

$\exists$ sequence of smooth blow ups $f: X \to Z$

of the strata of B s.t. $f_* \Delta \subseteq B \}$

Then $\exists$ sequence of blow ups $Y \to Z$ of the strata of B

s.t. $\forall (X, \Delta) \in \mathcal{F}$: $\mathrm{vol}(Y, K_Y + \Gamma) = w$

($\Gamma :=$ (strict transform of Δ) + (exep'l divs of $Y \dashrightarrow X$))

pf. $\boxed{\text{Idea}}$ Choose Y well $\Rightarrow \mathrm{vol}(K_Y + a\Gamma) \leq \mathrm{vol}(K_X + \Delta) = w$ for $a < 1$

Then use DCC in volumes!

$n = \dim Z$, assume $1 \in I$.

$\mathcal{G} := \{ (Y, \Gamma)$ log smooth $\mid Y$ proj dim n, coeff's of $\Gamma \in I \}$

([AL], (1.2)) $V = \{ \mathrm{vol}(Y, K_Y + \Gamma) \mid (Y, \Gamma) \in \mathcal{G} \}$ satisfies DCC.

$\Rightarrow \exists \delta > 0$: $\mathrm{vol}(Y, K_Y + \Gamma) \leq w + \delta \Rightarrow \mathrm{vol}(Y, K_Y + \Gamma) \leq w$.

([AL], (7.3)) $\exists r \in \mathbb{N}$: $\forall (Y, \Gamma) \in \mathcal{G}$: $K_Y + \Gamma$ big $\Rightarrow K_Y + \frac{r-1}{r}\Gamma$ big.

Pick $\varepsilon > 0$ s.t. $(1-\varepsilon)^n > \frac{\omega}{\omega+\delta}$ and set $a = 1 - \frac{\varepsilon}{r} < 1$.

$$K_Y + aT = (1-\varepsilon)(K_Y + T) + \varepsilon\left(K_Y + \frac{r-1}{r}T\right)$$

so
$$\mathrm{vol}(K_Y + aT) \geq (1-\varepsilon)^n \,\mathrm{vol}(K_Y + T) \geq \frac{\omega}{\omega+\delta}\,\mathrm{vol}(K_Y+T).$$

<u>Construction of $g: Y \to Z$</u>

Since (Z, aB) klt $\Rightarrow$ can apply (2.8.1)!

$\exists\, g: Y \to Z$ s.t. $K_Y + \Psi_0 = g^*(K_Z + aB) + E_0$

$$\begin{cases} \Psi_0 \geq 0,\ E_0 \geq 0 & \text{no common components} \\ g_*\Psi_0 = aB,\ g_*E_0 = 0 \end{cases}$$

$\Rightarrow$ no two components of Ψ_0 intersect.

In particular, (Y, Ψ_0) terminal.

Pick $(X, \Delta) \in \mathcal{F}$. $T = $ (strict transform of Δ) + (excep'l divs of $h: Y \dashrightarrow X$)

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ Y \arrow[ur, dashed, "h"] \arrow[rr, "g"'] & G & Z \end{tikzcd}$$

Δ (on X), T (on Y), Φ (on Z)

$\Phi := g_*(aT) = a f_*\Delta \leq aB.$

~~(Z, Φ) klt $\Rightarrow$ apply (2.8.1)~~

✓ $K_Y + \Psi = g^*(K_Z + \Phi) + E$

$$\begin{cases} \Psi \geq 0,\ E \geq 0 & \text{no common components} \\ g_*\Psi = \Phi,\ g_*E = 0. \end{cases}$$

Since $\Phi \leq aB$, $\Psi \leq \Psi_0$. $\Rightarrow (Y, \Psi)$ terminal.

Recall [BA] Lemma 5.3.(2)

(X, Δ) proj. log pair, X, Y $\mathbb{Q}$-factorial.

$X \xrightarrow{\pi} Y$ bir'l, $\Theta = \Delta \wedge \underbrace{L_{\pi_*\Delta, X}}$, then

$\mathrm{vol}(K_X + \Delta) = \mathrm{vol}(K_X + \Theta)$.

$$(K_X + L_{\pi_*\Delta, X} = \pi^*(K_Y + \pi_*\Delta) + E)$$

Let $\Theta := \Phi \wedge aT$, $\Sigma := $ (strict transform of Θ on X).

$\Rightarrow \Sigma \leq a\Delta \leq \Delta$.

$$\begin{aligned}\mathrm{vol}(Y, K_Y + aT) &= \mathrm{vol}(Y, K_Y + \Theta) \quad ([BA]\ (5.3.(2)))\\ &= \mathrm{vol}(X, K_X + \Sigma) \quad ((Y, \Theta) \text{ terminal})\\ &\leq \mathrm{vol}(X, K_X + \Delta) = w.\end{aligned}$$

$$w \leq \mathrm{vol}(Y, K_Y + T) \leq \frac{w+\delta}{w} \mathrm{vol}(Y, K_Y + aT) \leq w + \delta.$$

$\Rightarrow \mathrm{vol}(Y, K_Y + T) = w$.

$\square$

Lemma 7.2 $n \in \mathbb{Z}^t$, $w \in \mathbb{R}^t$, $I \subset [0,1]$ DCC set.

$\mathcal{F} := \{(X, \Delta)$ lc pairs $|$ X proj dim n, coeff's of $\Delta \in I$, $\mathrm{vol}(X, K_X + \Delta) = w\}$.

Then $\exists$ proj morphism $Z \to U$ & log smooth pair (Z, B) over U

s.t. $\forall (X, \Delta) \in \mathcal{F}$: $\exists u \in U$ and bir'l map $f_u : X \dashrightarrow Z_u$

such that $\mathrm{vol}(Z_u, K_{Z_u} + \Phi) = w$

$(\Phi := (\text{strict transform of } \Delta) + (\text{excep'l divs of } f_u^{-1}))$

pf. $\boxed{\text{Idea}}$

- $\mathcal{F}$ log birationally bounded $\Rightarrow$ take a prototype $Z \to U$
- some modifications $\Rightarrow$ prove the volume property for pairs (X_0, Δ_0) s.t. $\exists$ smooth blow ups $X_0 \to Z_0$ (use (7.1))
- use deformation invariance of volume ((4.3))

Assume $1 \in I$.

$\mathcal{F}$ is log birationally bounded!

$\Rightarrow$ $\exists$ proj morphism $Z \to U$ & log pair (Z, B) over U

s.t. $\forall (X, \Delta) \in \mathcal{F}$: $\exists u \in U$ and bir'l map $f_u : X \dashrightarrow Z_u$

such that $\mathrm{Supp}\,\Phi \subseteq \mathrm{Supp}\, B_u$.

Also, can assume $\begin{cases} (Z, B) \text{ log smooth over } U \\ \text{intersection of strata of } B \text{ w/ fibres is irreducible} \end{cases}$

([BA] proof of (1.9))

(fiber of Z at 0)

Fix closed point $0 \in U$.

$\mathcal{F}_0 = \{(X_0, \Delta_0) \in \mathcal{F} \mid \exists$ smooth blow ups $f : X_0 \to Z_0$ of the strata of B_0 with $f_* \Delta_0 \le B_0 \}$.

(7.1) $\Rightarrow$ $\exists$ sequence of blow ups $g : Y_0 \to Z_0$ of the strata of B_0 s.t. $\forall (X_0, \Delta_0) \in \mathcal{F}_0$: $\mathrm{vol}(Y_0, K_{Y_0} + \Gamma_0) = w$.

((strict transform of Δ_0) + (excep'l divs of $Y_0 \dashrightarrow X_0$))

Let $g : Y \to Z$ sequence of blow ups of the strata of B induced by g_0. $C :=$ (strict transform of B) + (excep'l divs of g).

$\mathcal{F}_1 = \{(X_0, \Delta_0) \in \mathcal{F} \mid \exists$ smooth blow ups $f : X_0 \to Y_0$ of the strata of C_0 with $f_* \Delta_0 \le C_0 \}$

$\mathcal{F}_1 \subseteq \mathcal{F}_0$. ($\exists f : X_0 \to Y_0 \Rightarrow X_0 \xrightarrow{f} Y_0 \xrightarrow{g_0} Z_0$)

$\forall (X_0, \Delta_0) \in \mathcal{F}_1$: $f_* \Delta_0 =$ (strict transform of Δ_0)

$\Rightarrow \mathrm{vol}(Y_0, K_{Y_0} + f_* \Delta_0) = w$

$$\begin{tikzcd} \Delta_0 \quad X_0 \arrow[rr, "(\mathrm{bir'l})", "f"'] \arrow[dr] & \circlearrowleft & Y_0 \arrow[dl, "g_0"] \\ & Z_0 & \end{tikzcd}$$

$\Rightarrow \mathrm{vol}(Z_0, K_{Z_0} + f_* \Delta_0) = w$.

Now replace

$\begin{cases} (Z, B) \text{ by } (Y, C) \\ \mathcal{F}_0 \text{ by } \mathcal{F}_1 \end{cases}$

Now I show this $Z \to U$ & (Z, B) works.

$\forall (X, \Delta) \in \mathcal{F}$: Pick $u \in U$ w/ $X \dashrightarrow Z_u$ bir'l.

By [BA] (proof of (1.9)), $\exists (X', \Delta')$ log smooth over Z_u s.t.

$$\begin{tikzcd} X \arrow[r, dashed, "\text{bir'l}"] \arrow[dr, dashed] \arrow[r, phantom, shift right=4, "\circlearrowleft"] & X' \arrow[d] \\ & Z_u \end{tikzcd}$$

and $X' \to Z_u$ blows up the strata of B_u

and $\operatorname{vol}(X, \underbrace{K_X + \Delta}_{(=w)}) = \operatorname{vol}(X', \underbrace{K_{X'} + \Delta'}_{(=w)})$.

Replace (X, Δ) by (X', Δ') $\Rightarrow$ assume (X, Δ) log smooth over Z_u & $X \to Z_u$ blows up the strata of B_u.

Let $h : W \to Z$ blow up the corresponding strata of B s.t. $W_u = X$, $h_u = f$.

$$\begin{tikzcd} & U & \\ \Theta \quad W \arrow[ur] \arrow[rr, "h"] \arrow[rr, phantom, shift left=5, "\circlearrowleft"] & & Z \arrow[ul] \\ \Theta_u = \Delta \quad W_u \simeq X \arrow[rr, "h_u = f"] & & Z_u \\ \Theta_0 \quad W_0 \arrow[rr, "h_0"] & & Z_0 \end{tikzcd}$$

Also $\Theta :=$ divisor on W such that $\Theta_u = \Delta$.

(4.3) $\Rightarrow \operatorname{vol}(W_0, K_{W_0} + \Theta_0) = \operatorname{vol}(W_u, K_{W_u} + \Theta_u)$

$= \operatorname{vol}(X, K_X + \Delta) = w$

$\Rightarrow (W_0, \Theta_0) \in \mathcal{F}_0$.

By conclusion in last page,

$\operatorname{vol}(Z_0, K_{Z_0} + h_{0*}\Theta_0) = w$.

$\| \; (4.3)$

$\operatorname{vol}(Z_u, K_{Z_u} + \Phi_u)$

$\Phi = h_*\Theta$.

$\Rightarrow \begin{cases} \Phi_0 = h_{0*}\Theta_0 \\ \Phi_u = (\text{strict transform of } \Delta) + (\text{excep'l div}) \end{cases}$

$\square$

Prop 7.3 Fix an integer n, a constant d, $I \subset [0,1]$ DCC set.

$$\mathcal{F}_{lc}(n, d, I) = \left\{ (X, \Delta) \;\middle|\; \begin{array}{l} X = \text{union of proj vars of dim } n \\ (X, \Delta) \text{ log canonical} \\ \text{coeff's of } \Delta \in I \\ K_X + \Delta \text{ ample } \mathbb{Q}\text{-divisor} \\ (K_X + \Delta)^n = d \end{array} \right\}$$

is bounded.

In particular, $\exists$ finite I_0 s.t. $\mathcal{F}_{lc}(n, d, I_0) = \mathcal{F}_{lc}(n, d, I)$.

pf. For any pair $(X, \Delta) \in \mathcal{F}_{lc}(n, d, I)$, let $X = \bigsqcup_{i=1}^{k} X_i$

and (X_i, Δ_i) corresponding lc pair.

$$\Rightarrow \begin{cases} K_{X_i} + \Delta_i \text{ is ample} \\ d_i = (K_{X_i} + \Delta_i)^n \end{cases} \Rightarrow d = \sum d_i.$$

There are only finitely many tuples $(d_1, \cdots, d_k)$,

and it suffices to show the set $\mathcal{F} \subseteq \mathcal{F}_{lc}(n, d, I)$

of irreducible pairs is bounded.

(7.2) $\Rightarrow$ $\exists$ proj morphism $Z \to U$ and a log smooth pair (Z, B) over U

s.t. $\forall (X, \Delta) \in \mathcal{F}$: $\exists\, u \in U$ and a birational map $f_u : Z_u \dashrightarrow X$ such that $\mathrm{vol}(Z_u, K_{Z_u} + \underset{\sim}{\Phi}) = d$.

$\underset{\sim}{\Phi}$ = ((strict transform of Δ) + (f_u-excep'l divisors))

(2.2.2.) $\Rightarrow$ f_u is the log canonical model of (Z_u, Φ).

(1.3) $\Rightarrow$ Replace U by a finite disjoint union of l.c. subsets and then assume every fiber of the morphism has a log canonical model.

Lastly, replace (Z, B) by the log canonical model over U

$\Rightarrow$ the fibers of the morphism are the elements of $\mathcal{F}$. $\square$

Theorem 1.1 / Main Result.

Fix an integer n, a constant d, $I \subset [0,1]$ DCC set.

$$\mathcal{F}_{slc}(n, d, I) = \left\{ (X, \Delta) \;\middle|\; \begin{array}{l} X = \text{union of proj vars of dim } n \\ (X, \Delta) \text{ slc} \\ \text{coeff's of } \Delta \in I \\ K_X + \Delta \text{ ample } \mathbb{Q}\text{-divisor} \\ (K_X + \Delta)^n = d \end{array} \right\}$$

is bounded.

In particular, $\exists$ finite I_0 s.t. $\mathcal{F}_{slc}(n, d, I_0) = \mathcal{F}_{slc}(n, d, I)$.

pf.

$$\mathcal{F}_{slc}(n, d, I) \xleftrightarrow{1:1} T = \left\{ \begin{array}{l} \text{Triples } (X, \Delta, \tau) \text{ where} \\ \cdot\ (X, \Delta) \in \mathcal{F}_{lc}(n, d, I) \\ \cdot\ \tau : S \to S \text{ is an involution} \\ \quad \text{of the normalisation of a} \\ \quad \text{divisor supported on } \lfloor \Delta \rfloor \\ \quad \text{which fixes the Diff} = \\ \quad (K_X + \Delta)|_S - K_S \end{array} \right\}$$

(7.3) $\Rightarrow$ $\mathcal{F}_{lc}(n, d, I)$ bounded

$\Rightarrow$ T bounded $\Rightarrow$ $\mathcal{F}_{slc}(n, d, I)$ bounded. $\square$